Grid Independent Bounding Surfaces in the Material Point Method

Carter M. Mast

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Participants:
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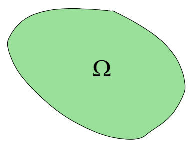
Outline

- **MPM Overview**
- **Grid Independent Bounding Surface**
- **Approaches**
- **Demonstrative Problem**
- **Summary and Conclusions**
- **Outlook/Future Research**

MPM Overview



- **Force equilibrium:**

$$div(\sigma) + \mathbf{b} - \rho\, \ddot{\mathbf{u}} = \mathbf{0} \quad \textbf{in} \quad \Omega$$

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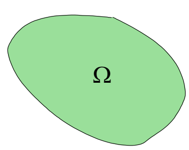
MPM Overview



- **Weak form:**

$$\int_\Omega (\sigma : \nabla \eta + \mathbf{b} \cdot \eta - \rho\, \mathbf{u} \cdot \eta)\, dV = 0$$

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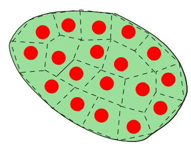
MPM Overview



$$\int_\Omega (\bullet)\; dV = \sum_P \int_{\Omega_P} (\bullet)\; dV_P$$

$$\int_\Omega (\boldsymbol{\sigma} : \nabla \boldsymbol{\eta} + \mathbf{b} \cdot \boldsymbol{\eta} - \rho\, \ddot{\mathbf{u}} \cdot \boldsymbol{\eta})\, dV = 0$$

MPM Overview

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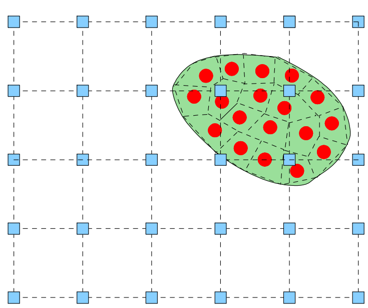


$$\int_\Omega (\bullet)\; dV = \sum_P \int_{\Omega_P} (\bullet)\; dV_P$$

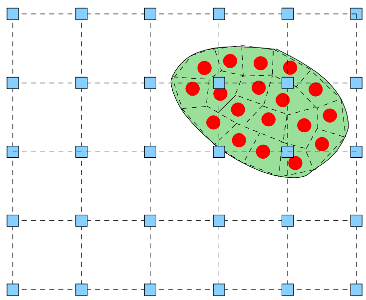
$\eta\,(x) = \sum_J N_J(x)\, \eta_J$

$\mathbf{u}(x) = \sum_J N_J(x)\, U_J$

$\eta\,(x) = \delta\, \mathbf{u}\,(x) = \sum_J N_J(x)\, \delta\, U_J$

$[M_{IJ}]\, \{A_J\} = \{F_I\}$

MPM Overview



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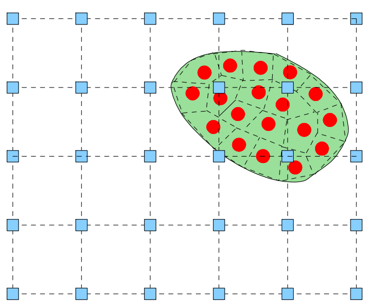

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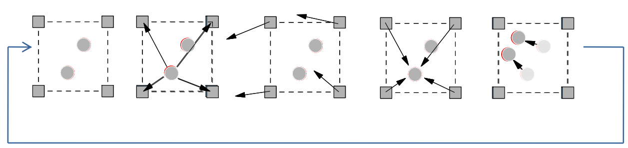
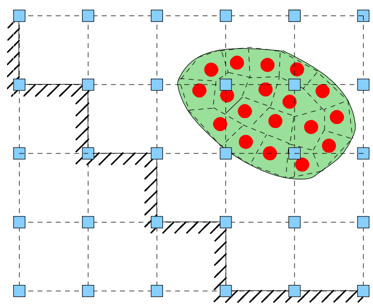


MPM Overview



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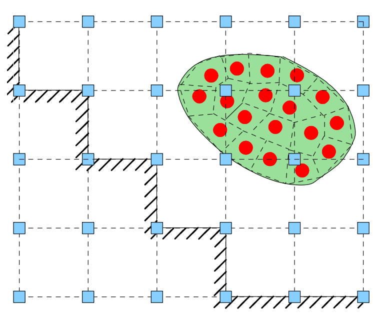
$$[M_{IJ}]\ \{A_J\} = \{F_I\}$$

$A_{1,2,3\ldots N} = 0$

How can a bounding surface be included?

MPM Overview



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$$[M_{IJ}]\ \{A_J\} = \{F_I\}$$

$A_{1,2,3\ldots N} = 0$

- **The current MPM algorithm allows for:**
  - **Horizontal bounding surfaces**
  - **Vertical bounding surfaces**
  - **Or combination thereof → Results in a "stepped" boundary**

MPM Overview

- Disadvantages of the current surface representation:
  - Bounding surfaces are *dependent* on the computational grid
  - Horizontal & Vertical boundaries are often unrealistic for the general essential boundary condition

The solution:

Introduce a grid independent bounding surface

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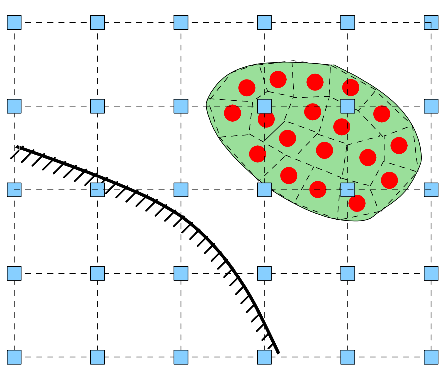
Grid Independent Surface

- Representation of a GENERAL surface—one that is independent of the background nodes—in the Material Point Method.



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MPM Overview

- Disadvantages of the current surface representation:
  - Bounding surfaces are *dependent* on the computational grid
  - Horizontal & Vertical boundaries are often unrealistic for the general essential boundary condition
- Advantages of a grid independent bounding surface:
  - General surfaces shapes can be represented
  - Allows for improved accuracy in determining the load distribution

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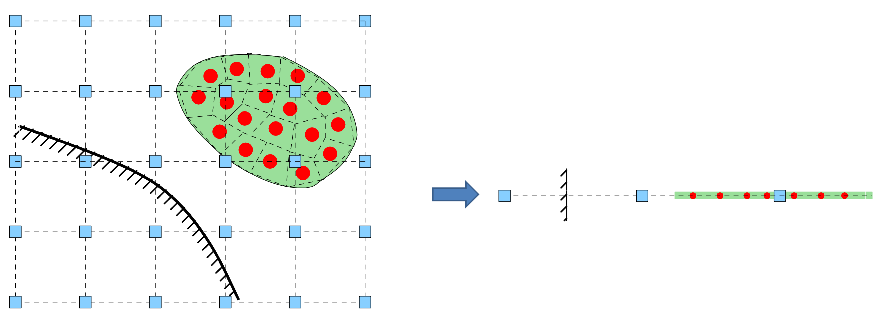
Incorporating the General Surface

- Develop a 1-D model



General Approach in 2-D or 3-D

1-D Representation

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Incorporating the General Surface

- Develop a 1-D model
- Approaches for incorporating the general surface:
  - "Ad-hoc" approaches:
    - Develop initial understanding/behavior of the problem
    - Use existing formulation
    - Find an "easy" solution to the problem

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Incorporating the General Surface

- Develop a 1-D model
- Approaches for incorporating the general surface:
  - "Ad-hoc" approaches:
    - Develop initial understanding/behavior of the problem
    - Use existing formulation
    - Find an "easy" solution to the problem
  - Variational approach:
    - Relies on the development of an enhanced weak form of the governing equation(s).
    - Use of the Lagrange multiplier method in which the bounding surface traction becomes the Lagrange multiplier

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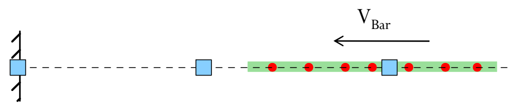
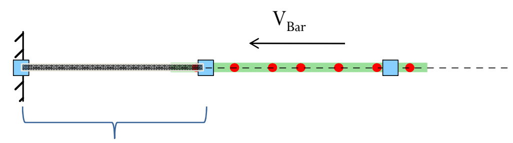
Incorporating the General Surface

- **Define necessary terms:**
  - **The idea of a boundary *length* or boundary *zone***
    - **Traditional MPM:**



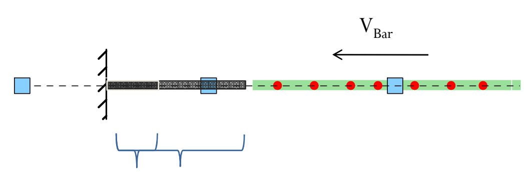
    - **Bar begins interaction with boundary when leading particle enters "boundary" cell:**



Boundary *length*, h

Incorporating the General Surface

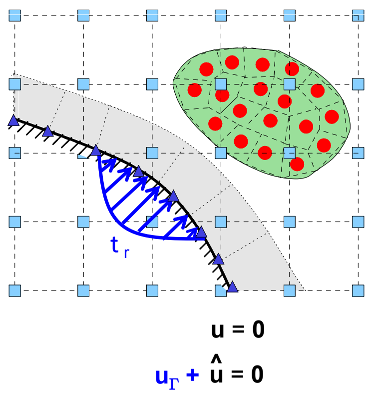
- **Define necessary terms:**
  - **The idea of a boundary *length* or boundary *zone***
    - **Traditional MPM:**
      - **Boundary length, h, is equal to the cell length**
    - **For the general surface MPM:**
      - **No relationship between boundary length and cell length (bounding surface is *independent* of the computational grid).**



Boundary *length*, h

Variational Approach

$$\int_\Omega (\sigma : \nabla \eta + \mathbf{b}\cdot\eta - \rho\,\ddot{\mathbf{u}}\cdot\eta)\, dV - \int_\Gamma \mathbf{t}_r \cdot \eta \, d\Gamma - \int_\Gamma (\hat{u} + u_\Gamma)\cdot \tau \, d\Gamma = 0$$



$\eta(x) = \sum_J N_J(x)\, \hat{\eta}_J + \eta_\Gamma$

$\mathbf{u}(x) = \sum_J N_J(x)\, \hat{U}_J + \mathbf{u}_\Gamma$

$\eta(x) = \delta\,\mathbf{u}(x) = \sum_J N_J(x)\, \delta\, \hat{U}_J + \delta\,\mathbf{u}_\Gamma$

where:

$\eta_\Gamma(x) = \sum_\beta N_\beta(x)\, \eta_\beta$

$\mathbf{u}_\Gamma(x) = \sum_\beta N_\beta(x)\, U_\beta$

$\delta\,\mathbf{u}_\Gamma(x) = \sum_\beta N_\beta(x)\, \delta\, U_\beta$

Variational Approach

$$\int_\Omega (\sigma : \nabla \eta + \mathbf{b}\cdot\eta - \rho\,\ddot{\mathbf{u}}\cdot\eta)\, dV - \int_\Gamma \mathbf{t}_r \cdot \eta \, d\Gamma - \int_\Gamma (u + u_\Gamma)\cdot \tau \, d\Gamma = 0$$

$$\begin{bmatrix} 0 & 1 & \{N_{J\Gamma}\}^T \\ 1 & [M_{\Gamma\Gamma}] & [M_{\Gamma J}]^T \\ \{N_{I\Gamma}\} & [M_{\Gamma I}] & [M_{IJ}] \end{bmatrix} \begin{Bmatrix} \{-f_r\} \\ \{A_\Gamma\} \\ \{A_J\} \end{Bmatrix} = \begin{Bmatrix} \{\tilde{a}_\Gamma\} \\ \{F_\Gamma\} \\ \{F_J\} \end{Bmatrix}$$

Variational Approach

$$\int_\Omega (\sigma : \nabla\eta + \mathbf{b}\cdot\eta - \rho\,\ddot{\mathbf{u}}\cdot\eta)\,dV - \int_\Gamma \mathrm{t}_r\cdot\eta\,d\Gamma - \int_\Gamma (\mathrm{u}+\mathrm{u}_\Gamma)\cdot\tau\,d\Gamma = 0$$

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- **The above system is only valid when particles(s) are in the boundary zone.**

Variational Approach

$$\int_\Omega (\sigma : \nabla\eta + \mathbf{b}\cdot\eta - \rho\,\ddot{\mathbf{u}}\cdot\eta)\,dV - \int_\Gamma \mathrm{t}_r\cdot\eta\,d\Gamma - \int_\Gamma (\mathrm{u}+\mathrm{u}_\Gamma)\cdot\tau\,d\Gamma = 0$$

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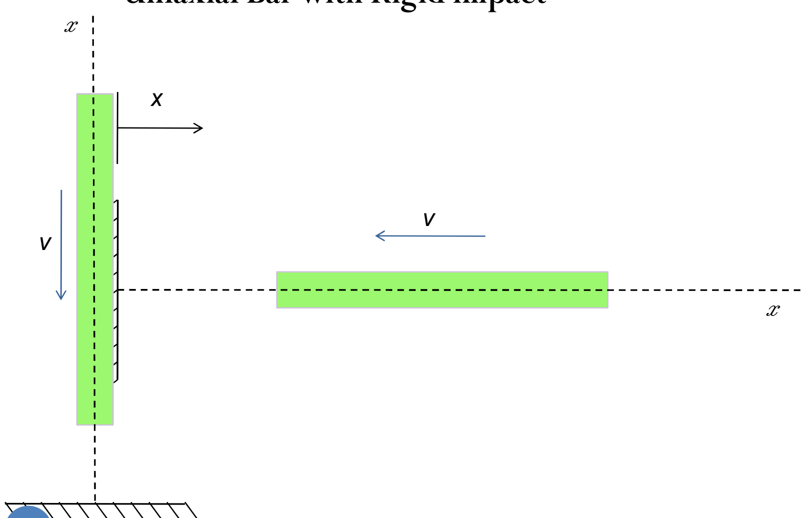
- **If all particles are outside of the boundary zone, traditional MPM is recovered.**

Demonstrative Problem

- **Uniaxial Bar with Rigid impact**
  - **Dynamic in Nature**
  - **Impact Problem**
  - **Known closed form solution**

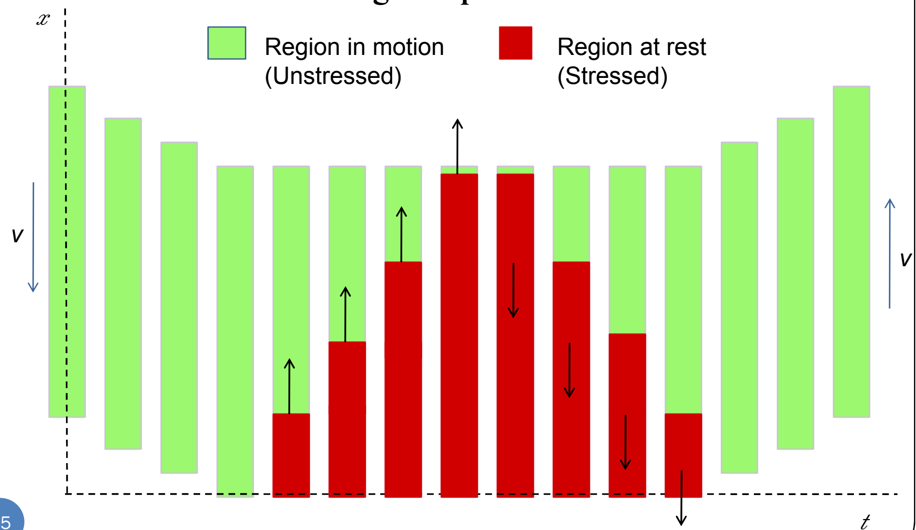
Demonstrative Problem

- **Uniaxial Bar with Rigid impact**



Demonstrative Problem

- **Uniaxial Bar with Rigid impact**



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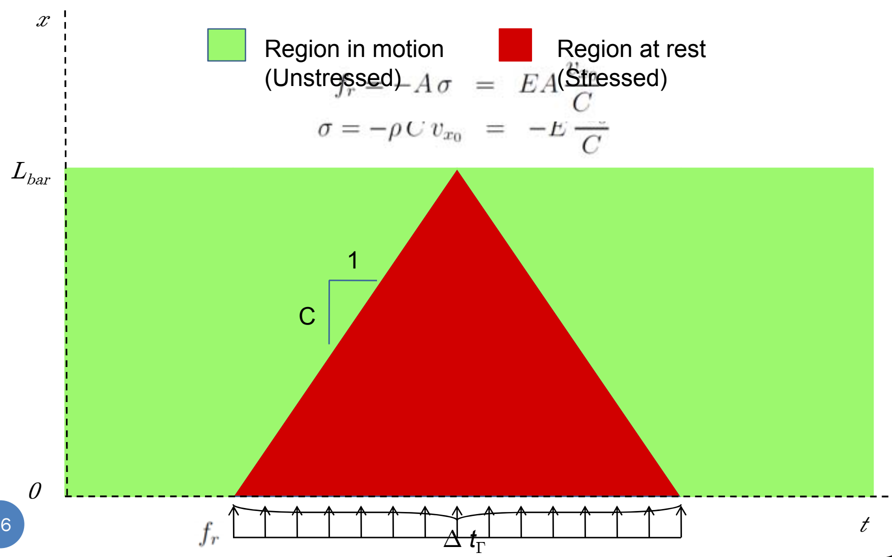
Demonstrative Problem

- **Visualization of stress**



$$f_r = -A\sigma = EA\frac{v_{x_0}}{C}$$
$$\sigma = -\rho C v_{x_0} = -E\frac{v_{x_0}}{C}$$

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Demonstrative Problem

- **Bar Properties**

Property	Variable	Value
Young's Modulus	$E_{bar} = E$	$29\,(10)^6$ psi
Mass Density	$\rho_{bar} = \rho$	$7.35\,(10)^{-4}$ lbf/in^3
Cross-sectional Area	$A_{bar} = A$	1.00 in^2
Bar Length	L_{bar}	50.00 in

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Demonstrative Problem

- **Bar Properties**

Property	Variable	Value
Initial Velocity	v_{x0}	-397.42 in/s
Wave Speed	C	$1.98\,(10)^5$ in/s
Time of Contact	Δt_Γ	$5.03\,(10)^{-4}$ s
Resulting Stress	σ	$-58,000$ psi
Reaction Force	f_r	$58,000$ lb

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Demonstrative Problem

- **MPM Setup**
  - **8 particles per cell**
  - **Cell length = 1"**
  - **Standard linear shape functions for both the background grid and boundary grid**
  - **Characterized boundary *length* as a fraction of the cell length**
  - **Characterized boundary *location* as a fraction of the cell length**

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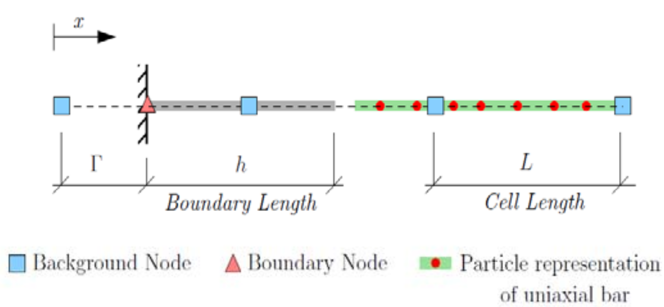
Demonstrative Problem

- **MPM Setup**
  - **General setup:**



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Demonstrative Problem

- **This particular study examined 55 representative cases**
- **Boundary location to cell length ratios of:**
  - **[0.0 - 1.0] by increments of 0.1**
- **Boundary length to cell length ratios of:**
  - **0.125, 0.25, 0.5, 0.75 and 1.0**
- **For example:**

$$\frac{\text{Boundary Length}}{\text{Cell Length}} = \frac{h}{L} = 0.125$$

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Demonstrative Problem

- **This particular study examined 55 representative cases**
- **Boundary location to cell length ratios of:**
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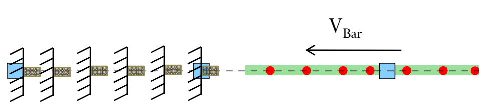

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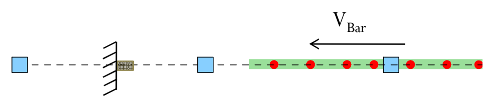
Demonstrative Problem

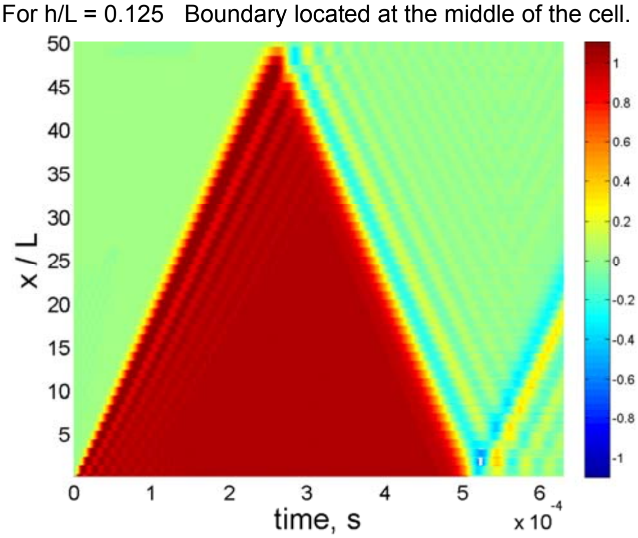
For h/L = 0.125 Boundary located at the middle of the cell.



Normalized Stress Plot

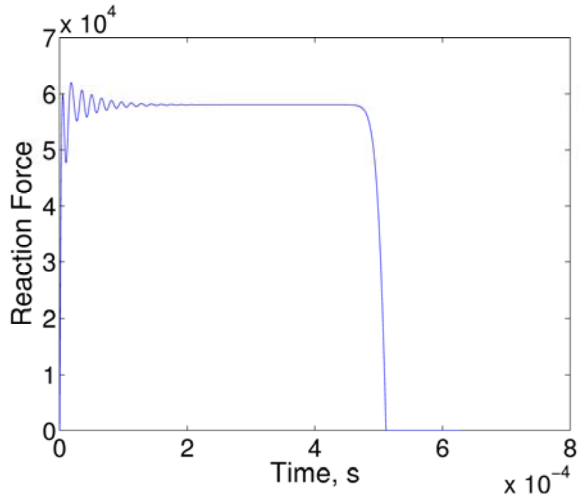
Demonstrative Problem

For h/L = 0.125 Boundary located at the middle of the cell.



Demonstrative Problem

For h/L = 0.125   Boundary located at the middle of the cell.



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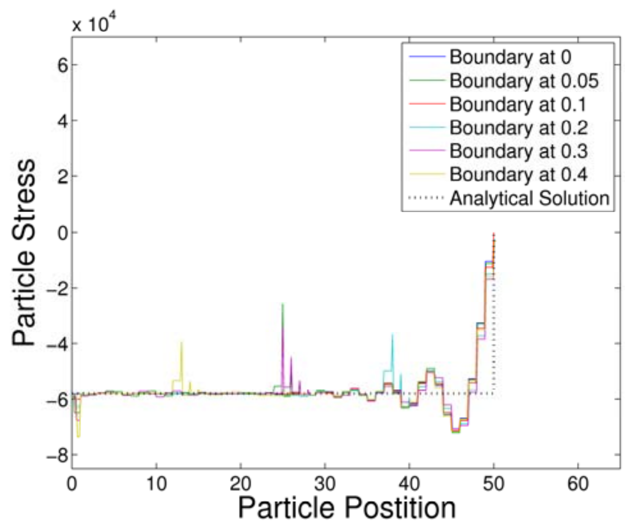
Demonstrative Problem



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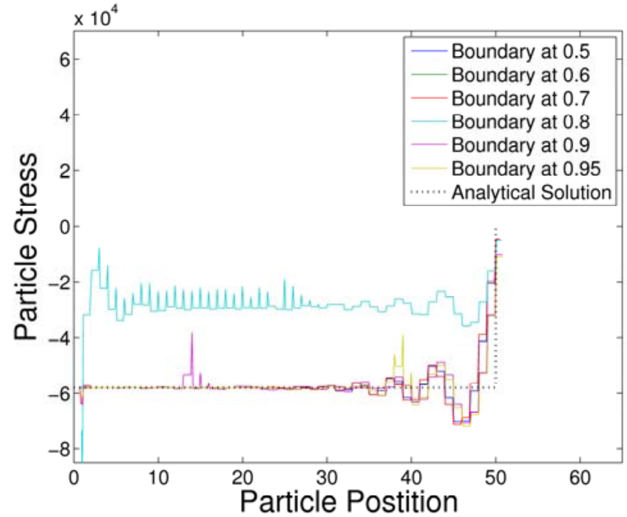
Demonstrative Problem



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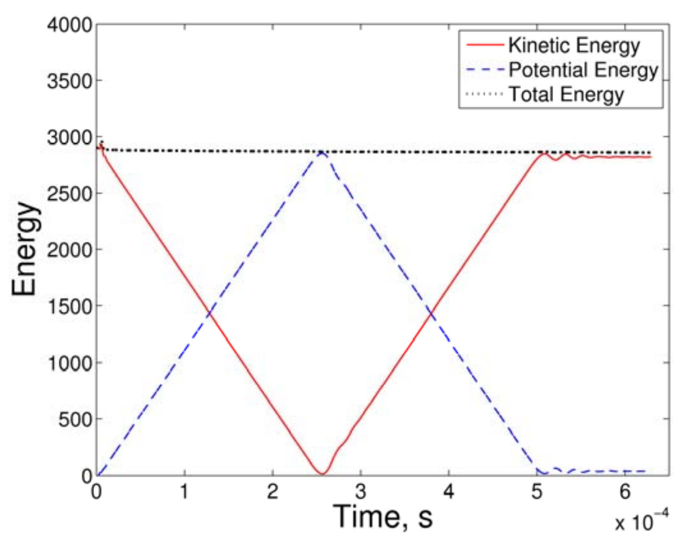
Demonstrative Problem



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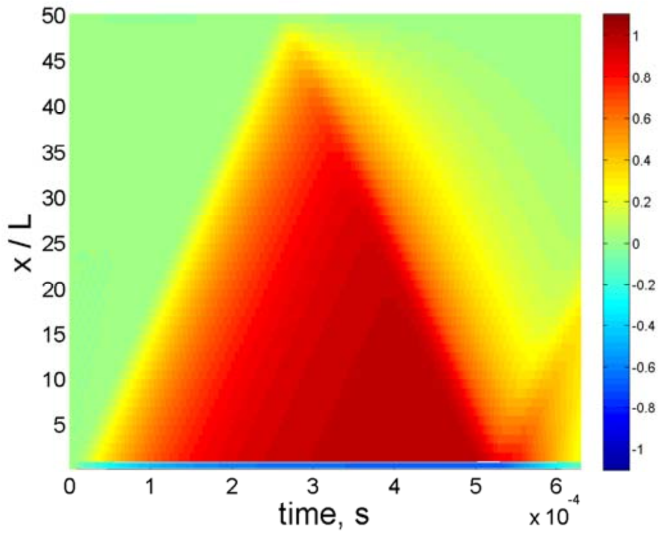
Demonstrative Problem

For h/L = 0.75   Boundary located at the middle of the cell.



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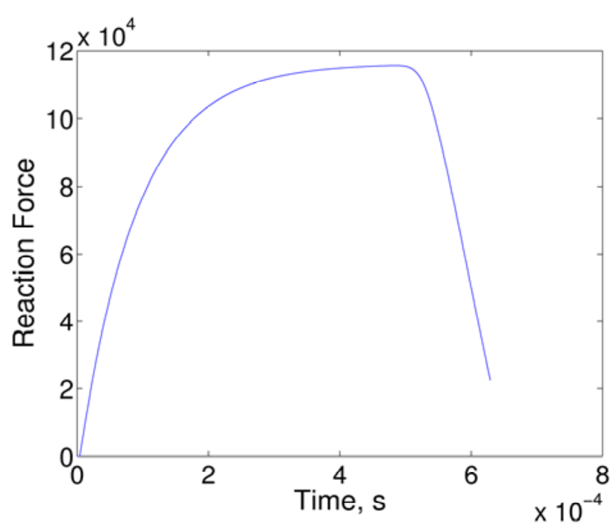
Demonstrative Problem



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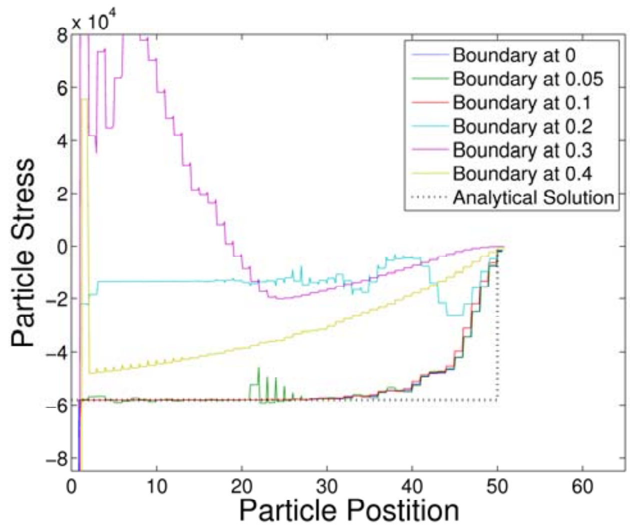
Demonstrative Problem



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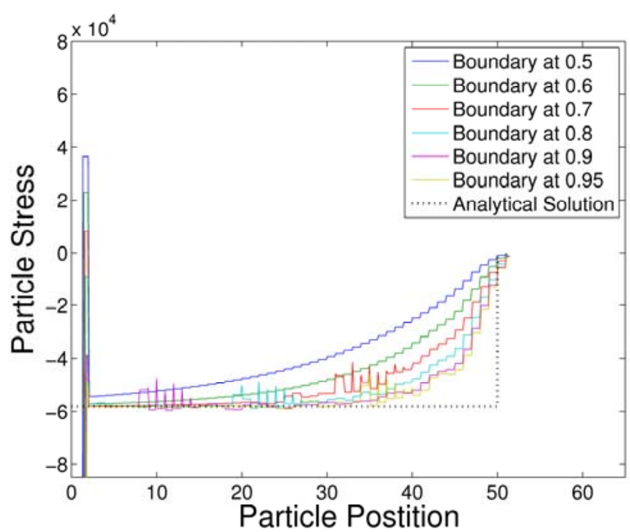
Demonstrative Problem



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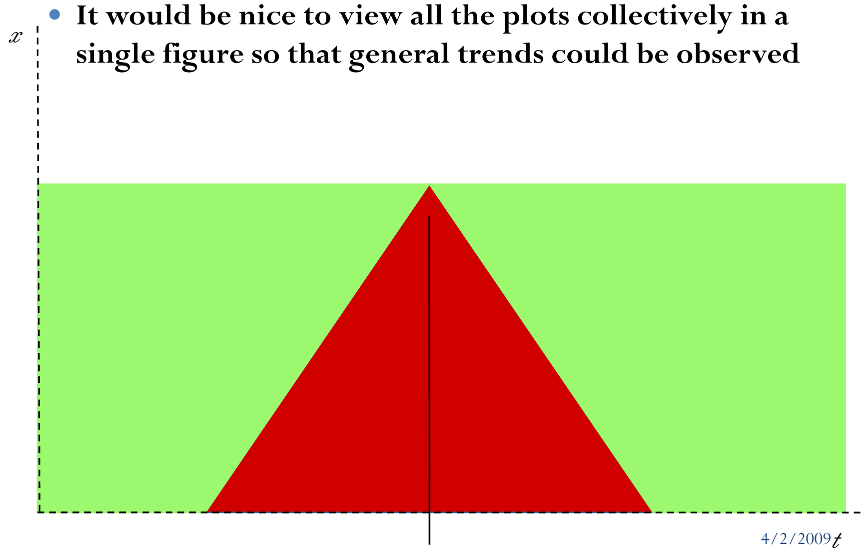
Demonstrative Problem

- It would be nice to view all the plots collectively in a single figure so that general trends could be observed



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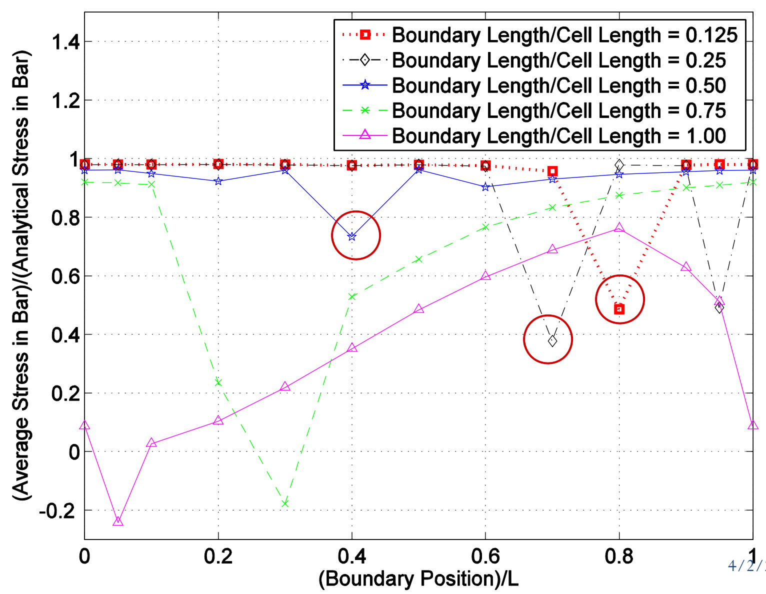

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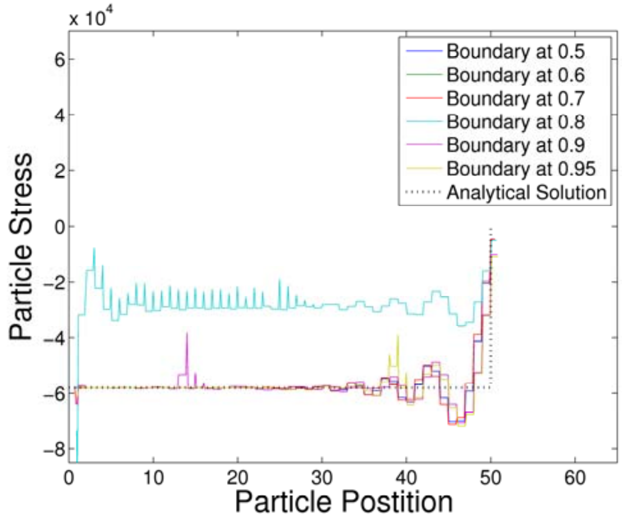

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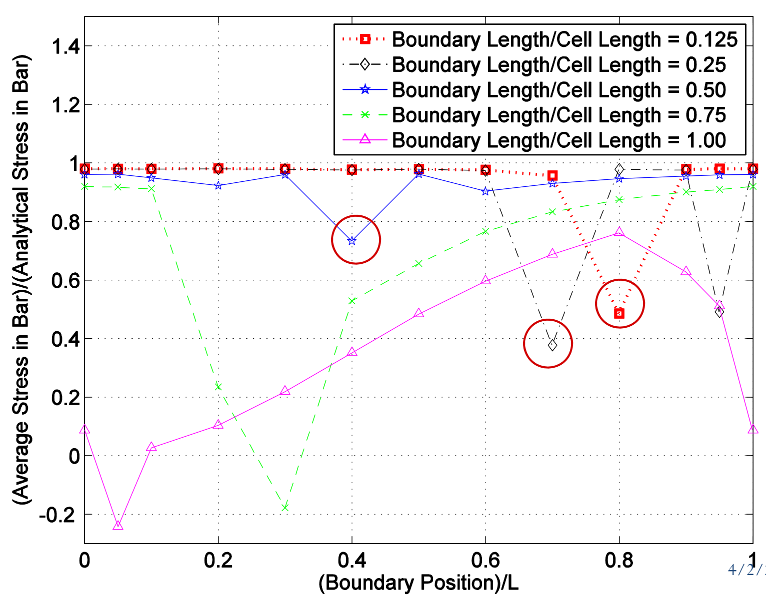
Demonstrative Problem



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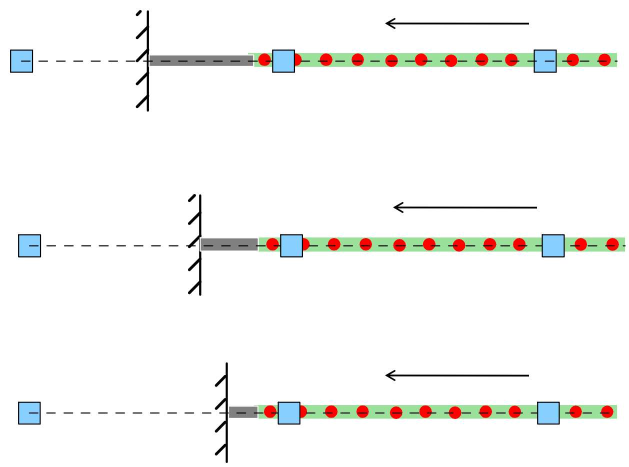
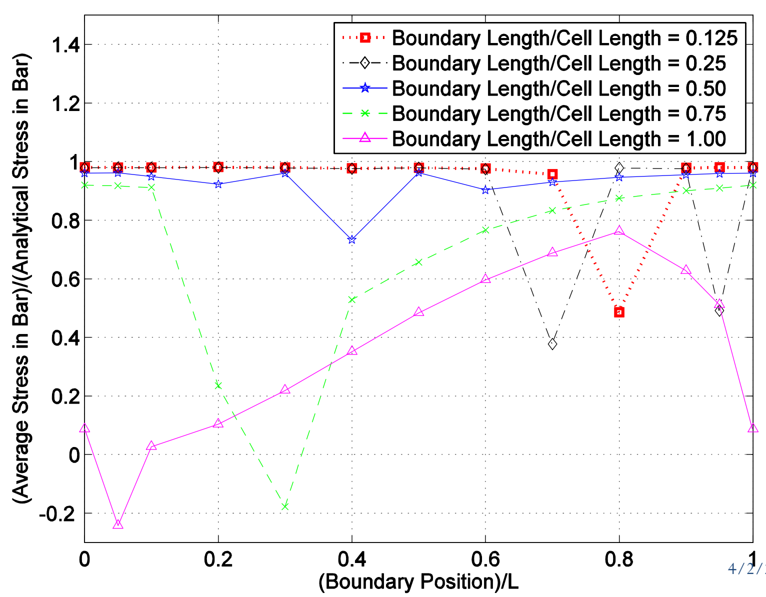
Demonstrative Problem



Demonstrative Problem



Summary and Conclusions

- It is possible to incorporate the general boundary condition into the MPM algorithm
- Findings in this study:
  - Shorter boundary lengths typically offer a more accurate or more desirable solution
  - Sensitivity within the algorithm to both:
    - Boundary location
    - Boundary length
  - If the boundary length is close to the cell, the resulting system of equations is singular or results in an ill-conditioned matrix eqn.
  - Issues with particle cell crossings persist and continue to have an impact on the results

Current Work

- Resolve issues with longer boundary lengths
  - Setting nodal acceleration for node behind the boundary to zero (effectively removes singularity)
  - Use of orthogonal shape functions between boundary grid and background grid (effectively removes singularity)
- Alleviate particle-cell crossing issues
  - Cell based averaging strategies
- Investigate the effects of using alternative shape functions for boundary field.
  - Preliminary studies show rather encouraging results
  - Use of orthogonal shape functions between boundary grid and background grid (effectively removes singularity)

Acknowledgments

- Greg Miller
- Peter Mackenzie
- Pedro Arduino
- Woo Kuen Shin

THANK YOU!

QUESTIONS?

Outline

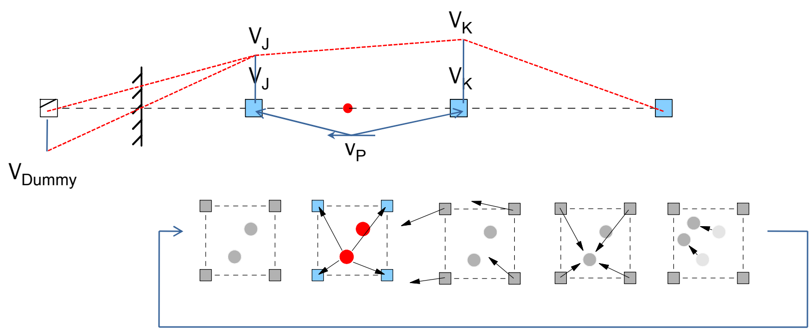
- MPM Overview
- Grid Independent Bounding Surface
- Approaches
- Demonstrative Problem
- Summary and Conclusions
- Outlook/Future Research
- **Additional figures and information for anyone who cares to see them or if there is time…**

Ad-hoc Approaches

- Objectives of the ad-hoc methods:
  - Develop an initial understanding of the problem
    - Look at MPM in 1-D
    - Effect of adding a boundary on the formulation
  - Use existing formulation
  - Find a "easy" solution to the problem
- Ad-hoc approaches were based primarily on imposing kinematic conditions
- Evaluated only a single particle

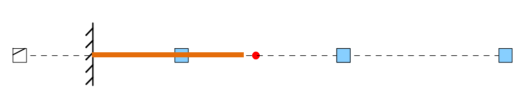
Ad-hoc Approaches

- **What do we mean by kinematic condition?**
  - $\mathbf{V}_{\Gamma} \cdot \mathbf{n} = 0$
  - $\Pi_{\Gamma} \cdot \mathbf{n} = 0$
- **In pictures (velocity used as an example):**



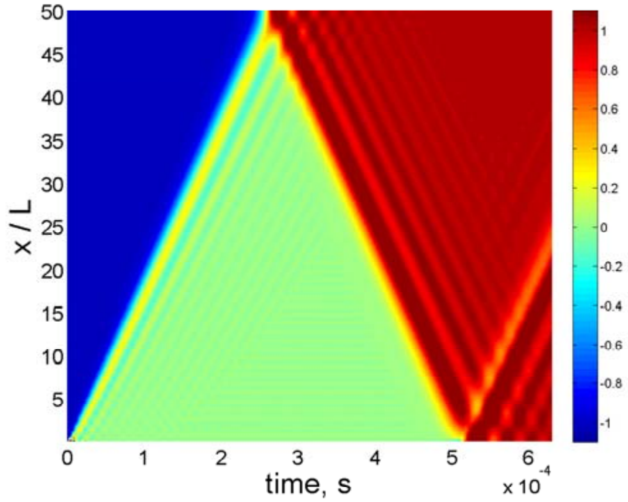
Ad-hoc Approaches

- **Overall did not lead to a desirable solution. However:**
  - **Provided valuable insight into the problem**
  - **Showed different trends**
    - **Particle going through the boundary**
    - **Particle sticking to boundary**
    - **Gaining excessive stress**
- **From the ad-hoc methods:**
  - **Idea of a boundary length of boundary zone introduced**



  - **Motivated additional study of the variational formulation**

Additional Figures



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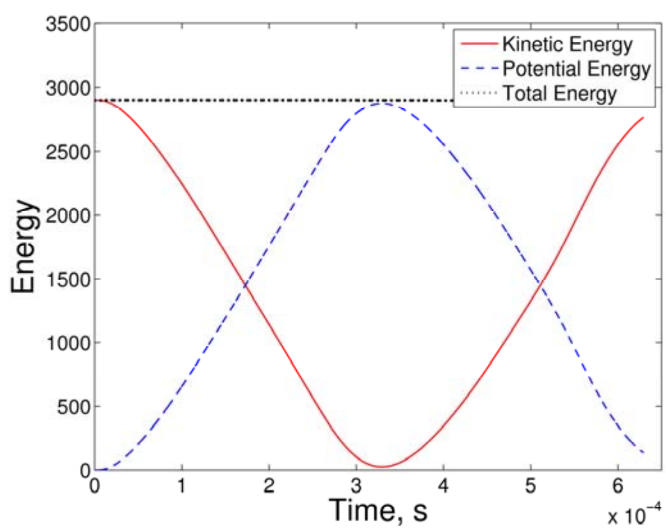
Additional Figures

Example of an energy plot: h/L = 0.75, boundary at 0.5



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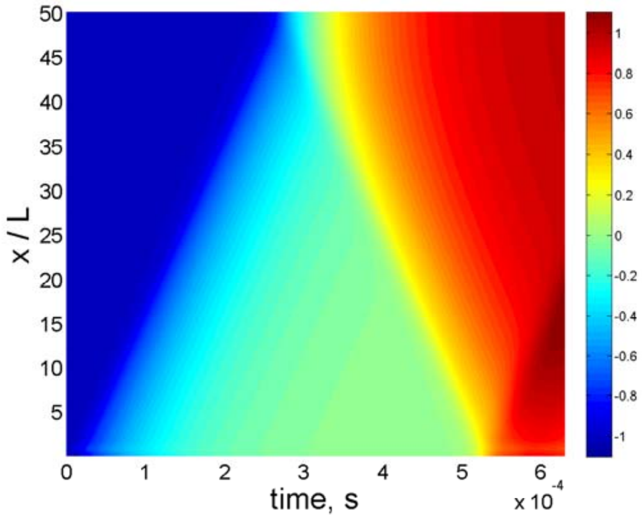
Additional Figures



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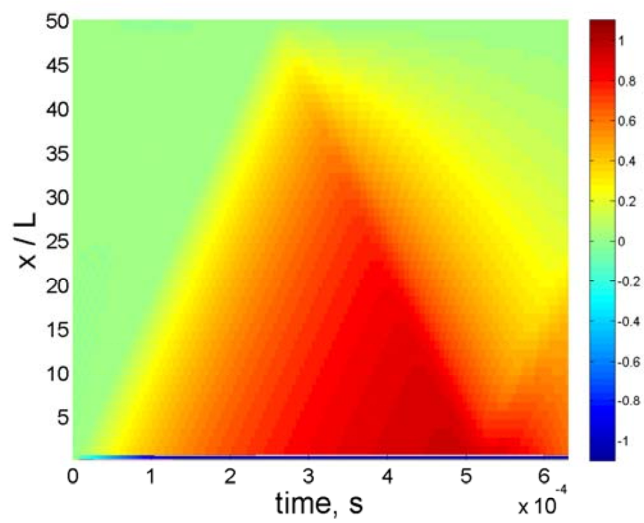
Additional Figures

Example of an energy plot: h/L = 1.0, boundary at 0.5



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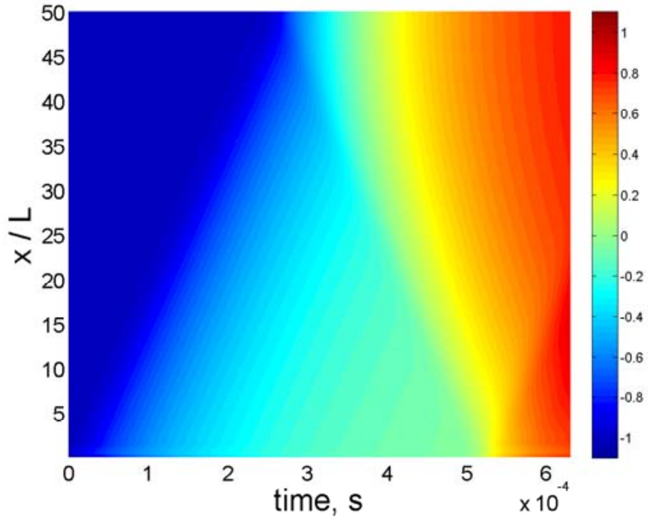
Additional Figures



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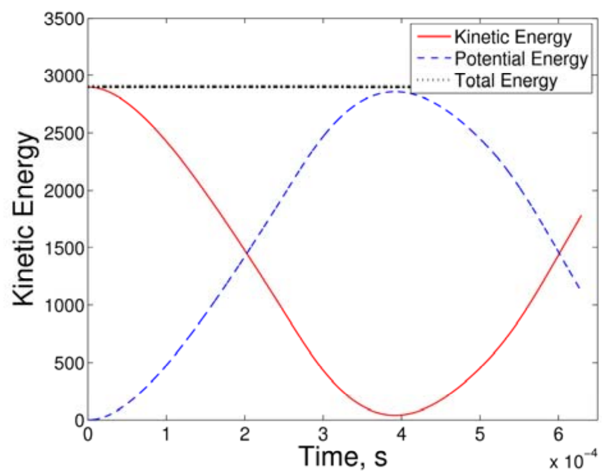
Additional Figures



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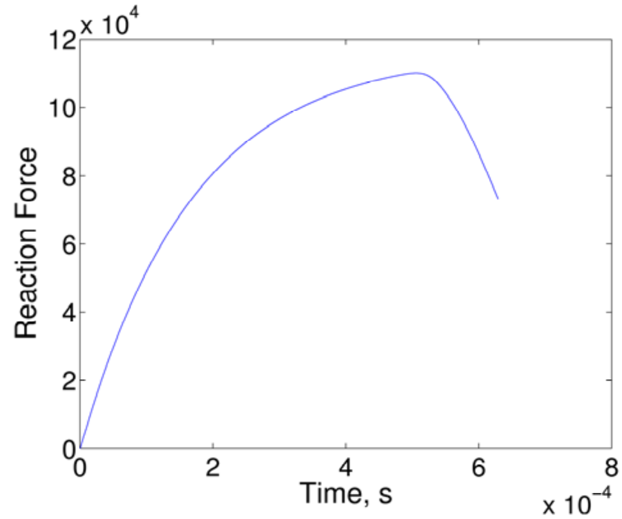
Additional Figures



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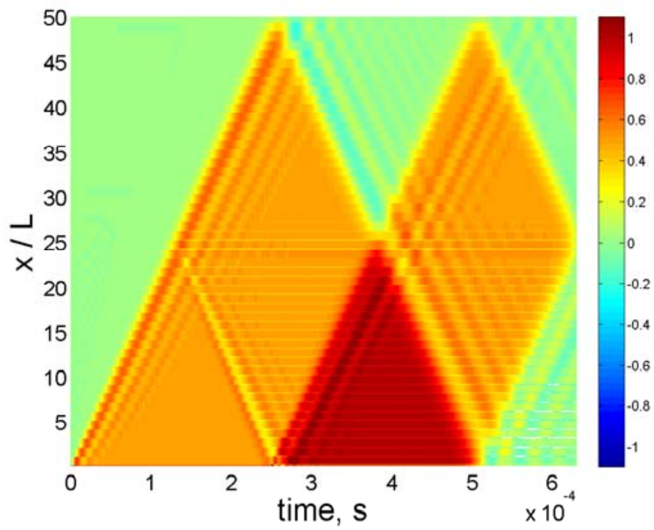
Additional Figures

Example of an energy plot: h/L = 0.125, boundary at 0.8



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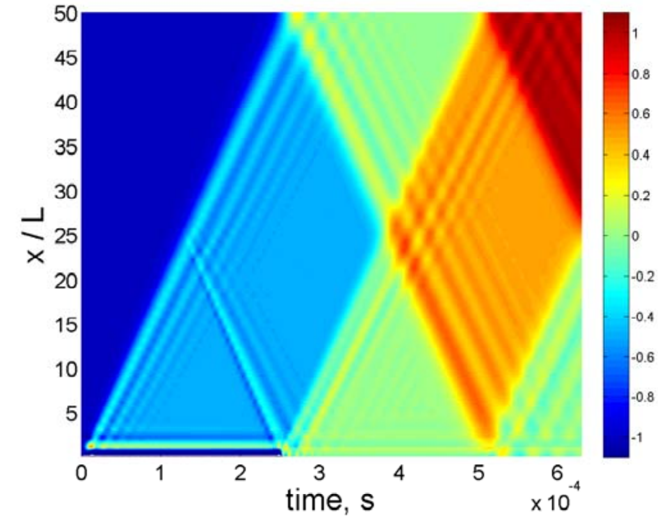
Additional Figures



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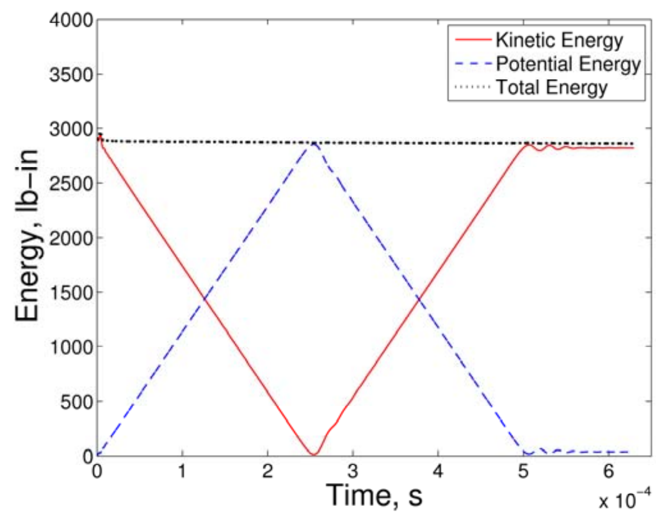

Additional Figures