Experiences with a Weighted Least Squares Material Point Method

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Outline and Summary

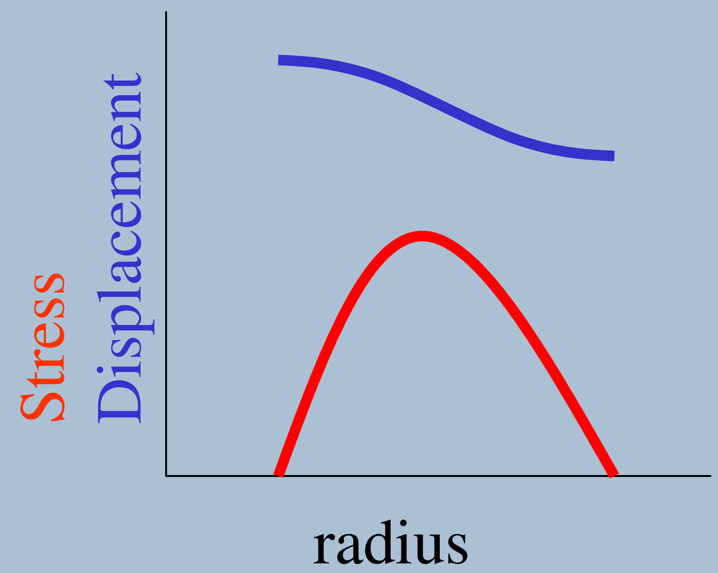
Local and global WLS, Quadrature, Implicit surface, Marching squares, Minimum weight trigger

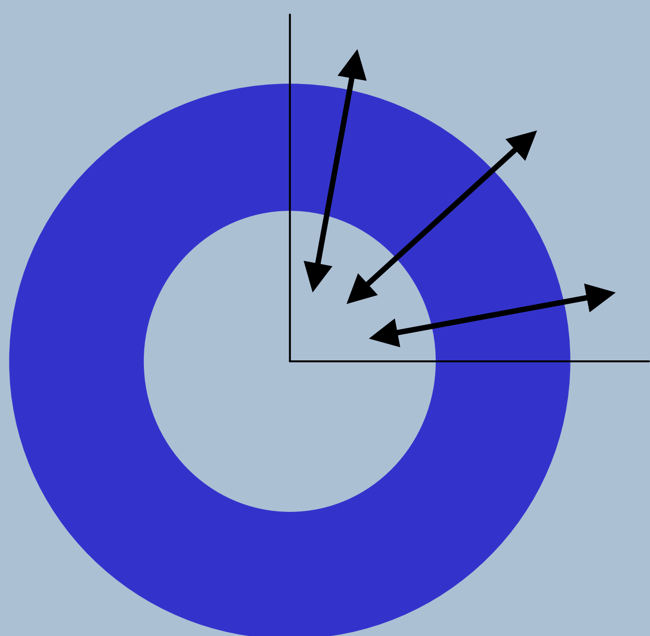
Finite element comparison code

Results for ring, ball, and channel

Expanding Ring

$$u(R,t) = T(t)\left[c_3 R^3 + c_2 R^2 + c_1 R\right]$$



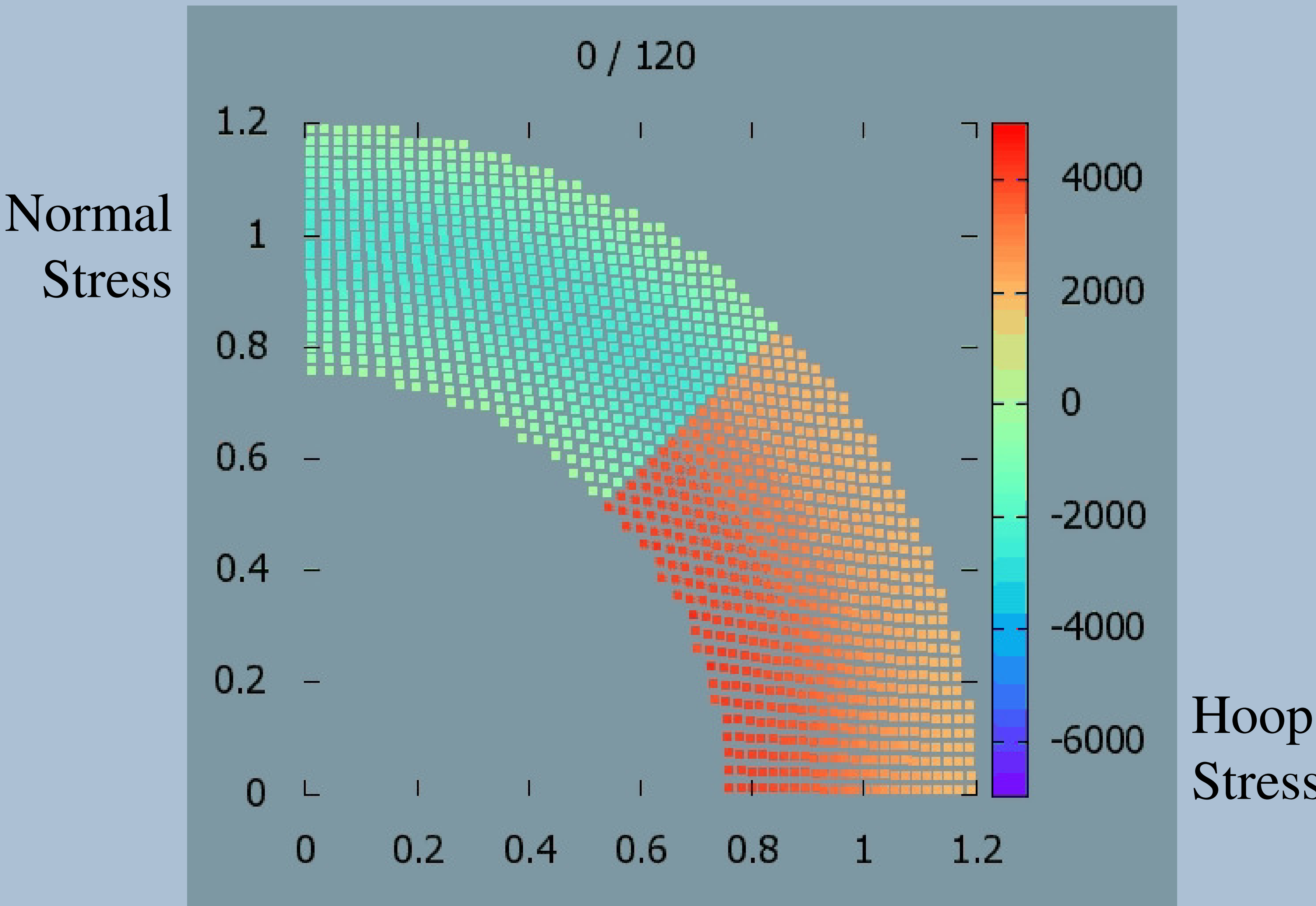


Free surfaces with implied zero normal stress

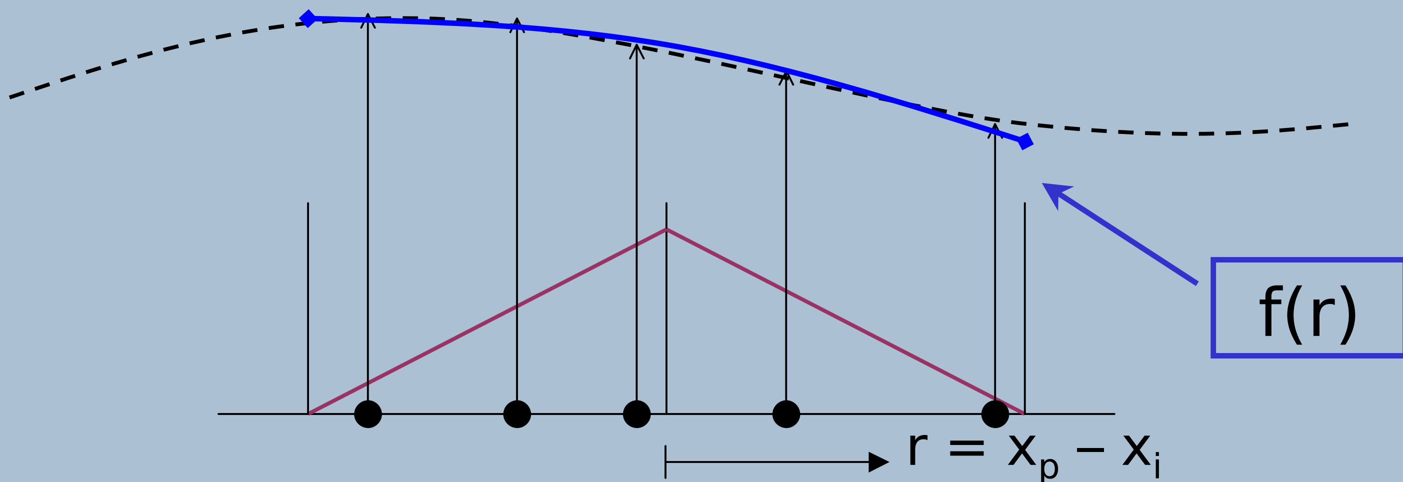
GIMP particle edges not aligned

More general and representative

Expanding Ring Results



Local Node Equation



$$f(r) = c_0 + c_1 r$$ Linear basis

$$f(r) = c_0 + c_1 r + c_2 r^2$$ Quadratic basis

Small Matrix Inversion

$$f(r) = c_0 + c_1 r = \mathbf{b} \cdot \mathbf{c} \qquad L_2 = \sum_p^n W(r_p)[f(r_p) - f_p]^2$$

$$\frac{\partial L_2}{\partial \mathbf{c}} \Rightarrow \begin{bmatrix} \sum S & \sum Sr \\ \sum Sr & \sum Sr^2 \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sum Sf \\ \sum Sfr \end{pmatrix}$$

- Small matrix on each node inverted once
- 1 mass, 6 stress, 3 momentum + other load vectors found with moment matrix
- Exact for functions in the basis

Local and Global Equations

Assemble local equation on a node from particles in the support:

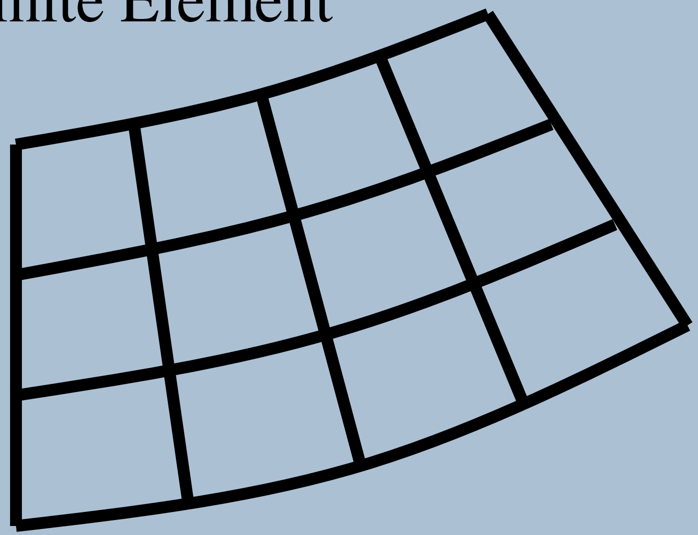
$$f_i^{\text{local}} = \mathbf{b}(r) \cdot \mathbf{c}_i = LS(f_p)$$

Assemble global equation from surrounding local node equations:

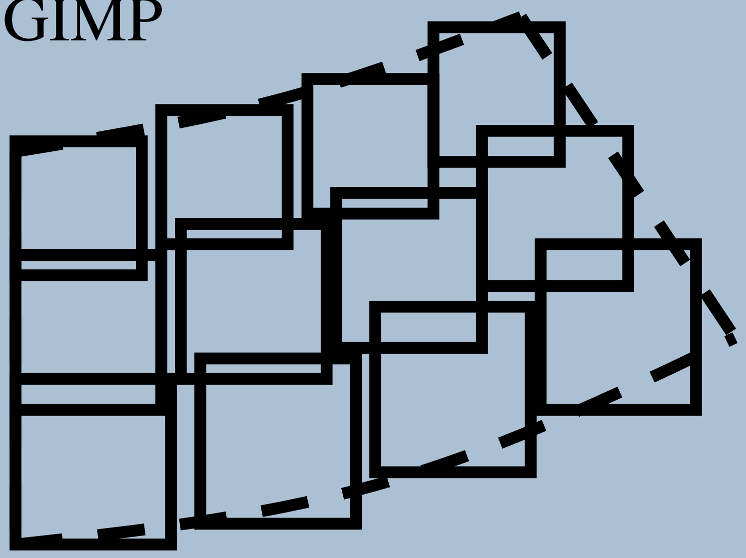
$$f(r) = \sum_i W_i(r)\mathbf{b}(r) \cdot \mathbf{c}_i$$

Volume Partitioning

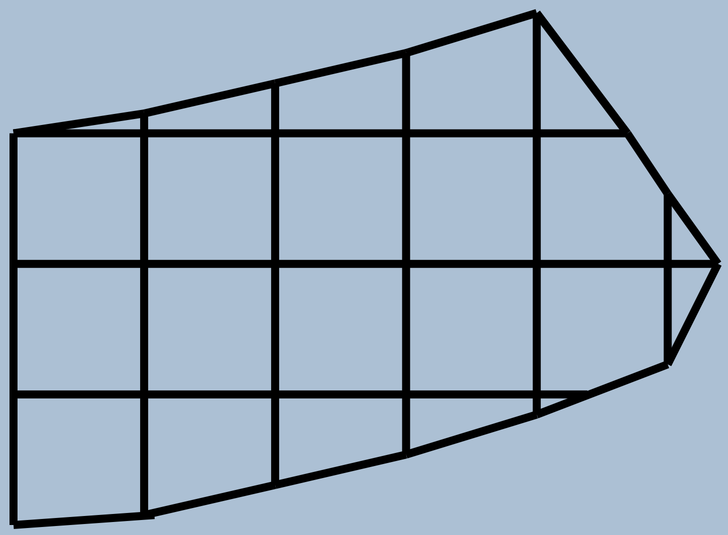
Finite Element



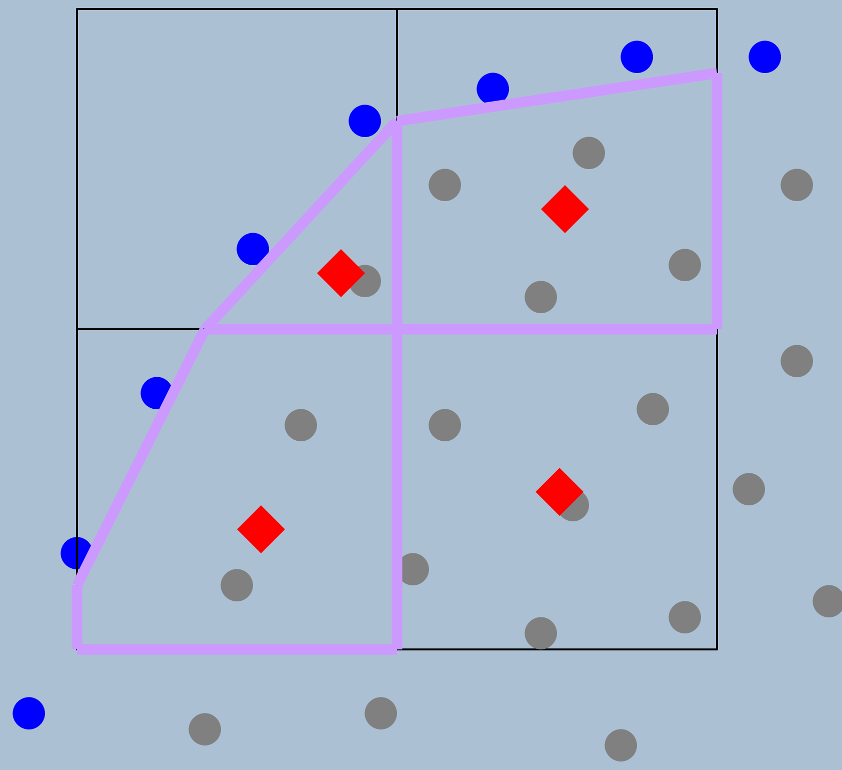
GIMP



Sub-divided cells



Single Point Gauss Quadrature at area centroid



Implicit surface

Integral of particle flags:

$$\beta_{\mathrm{i}} = \sum_{\mathrm{i}} \mathrm{S}_{\mathrm{ip}} \beta_{\mathrm{p}}$$

Each near-surface node finds the average position of surface particles visible to it.

$$\mathbf{x}_{\mathrm{i}}^{\mathrm{surf}} = \frac{\sum_{\mathrm{p}} \mathbf{x}_{\mathrm{p}}^{\mathrm{surf}} \mathrm{S}_{\mathrm{ip}}}{\sum_{\mathrm{p}} \mathrm{S}_{\mathrm{ip}}}$$

Surface normal is gradient of particle flags:

$$\mathbf{n} = \sum_{\mathrm{i}} \mathrm{G}_{\mathrm{i}} (\mathbf{x}_{\mathrm{i}}^{\mathrm{surf}} - \mathbf{x}_{\mathrm{i}}) \beta_{\mathrm{i}}$$

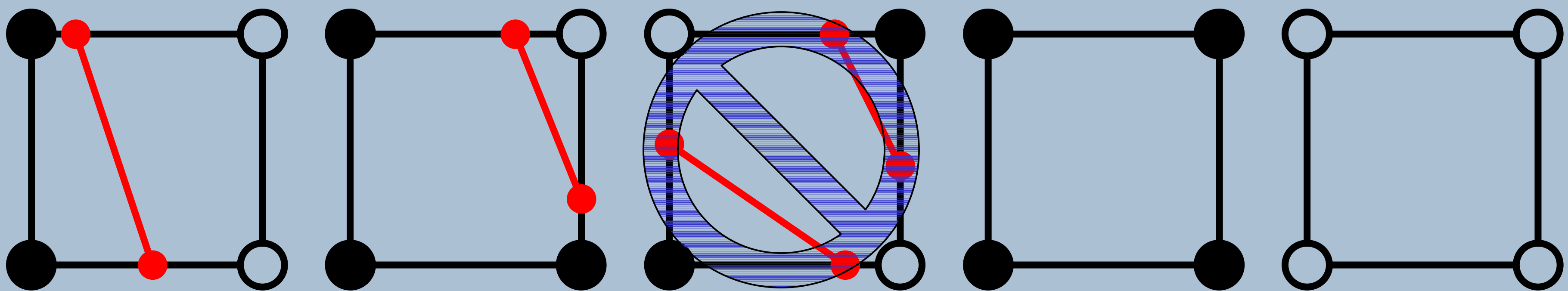
Signed distance is projection of local distance onto surface normal:

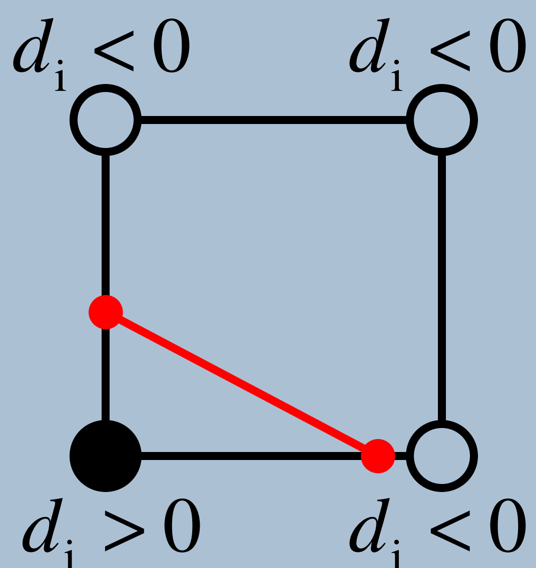
$$d_{\mathrm{i}} = -sign\left((\mathbf{x}_{\mathrm{i}}^{\mathrm{surf}} - \mathbf{x}_{\mathrm{i}}) \cdot \mathbf{n}\right) \left\| \mathbf{x}_{\mathrm{i}}^{\mathrm{surf}} - \mathbf{x}_{\mathrm{i}} \right\|$$

Marching Squares

Four nodes with two possible states = 2^4 = 16 possibilities.

With reflections and rotations we reduce the cases to:

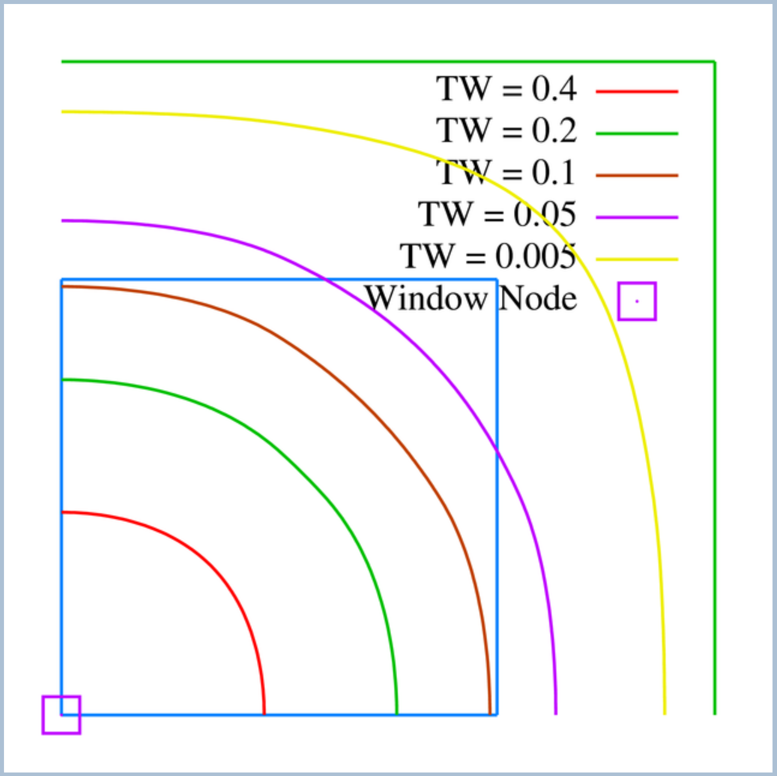




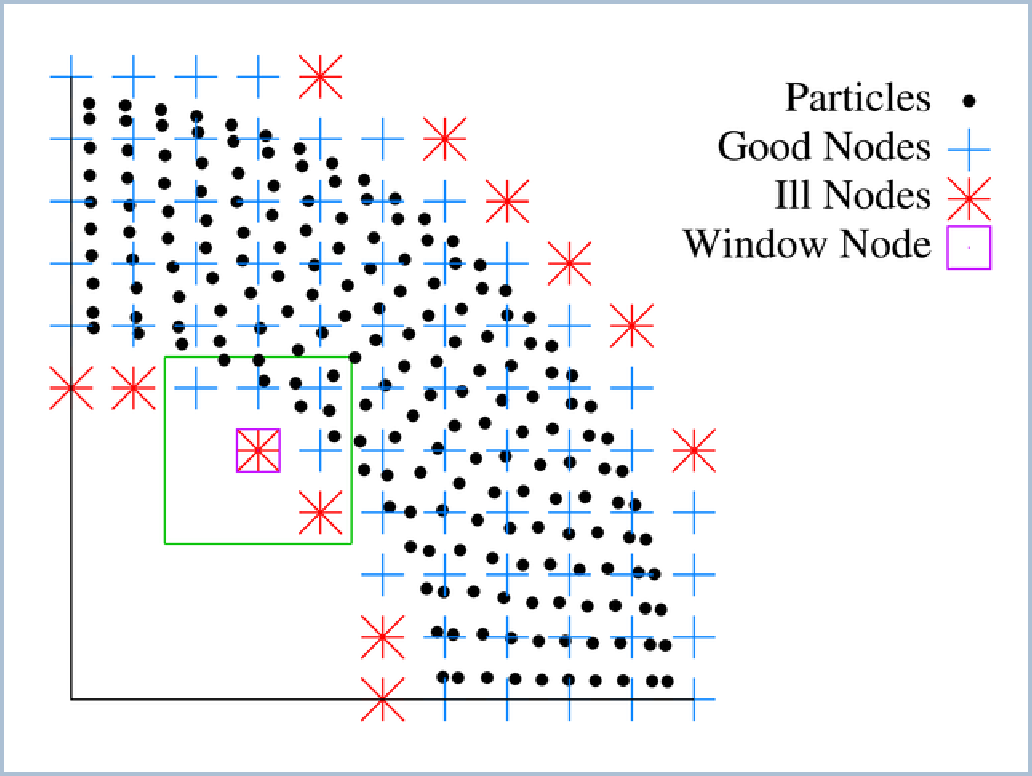
$$x_{cut} = x_I - d_I \frac{x_I - x_O}{d_I - d_O}$$

Minimum Weight Trigger

Trigger Weight



Ill-conditioned nodes



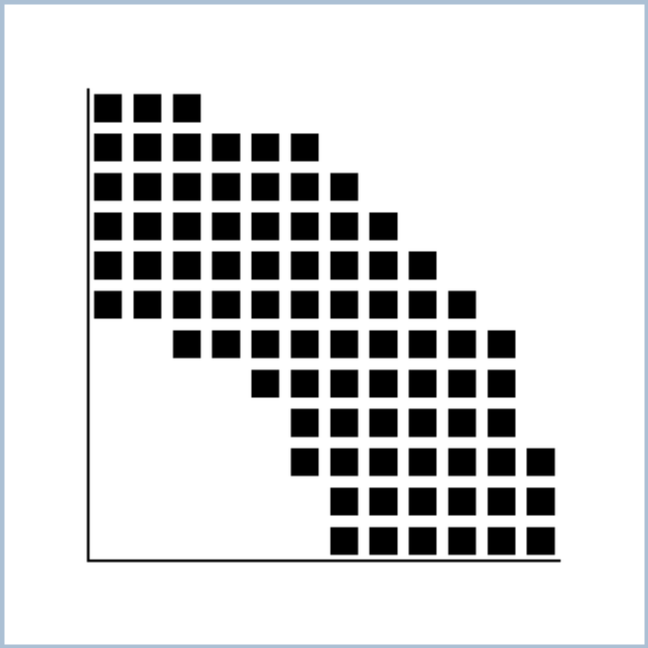
Rank reduction: $f_i^{LS} = \Sigma_p W_p f_p / \Sigma_p W_p$

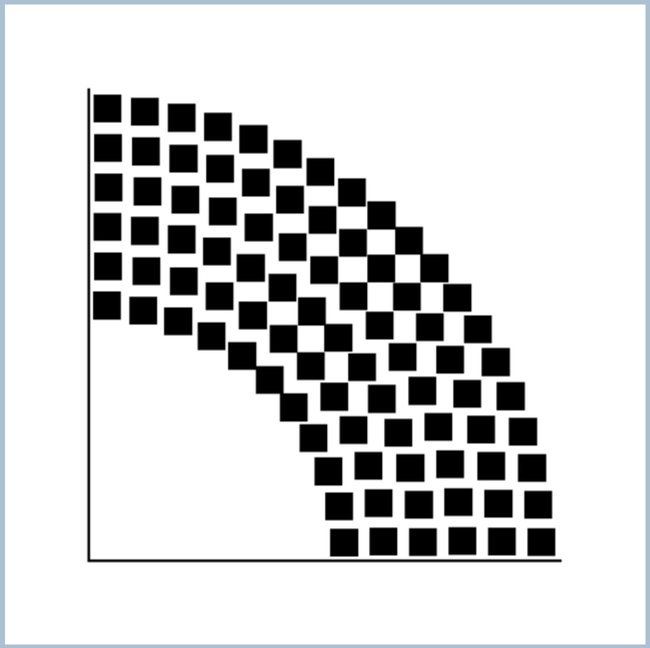
Neighbor averaging: $f_i^{LS} = \Sigma_k^{N_{good}} f_k^{LS} / N_{good}$

Particle Arrangements

Cartesian – surfaces approximated with stair-steps

Radial – better surface approximation, but more gaps and overlaps for interior





Comparison FEM code

Linear triangles with single gauss point integration

Discrete Momentum:

$$a_i = \frac{f_i^{int}}{M_i} + f_i^{ext}(X_i, t)$$

Mass lumping based on reference configuration:

$$M_i = \frac{1}{3}\rho^0 \sum_e A_e^0$$

Internal forces:

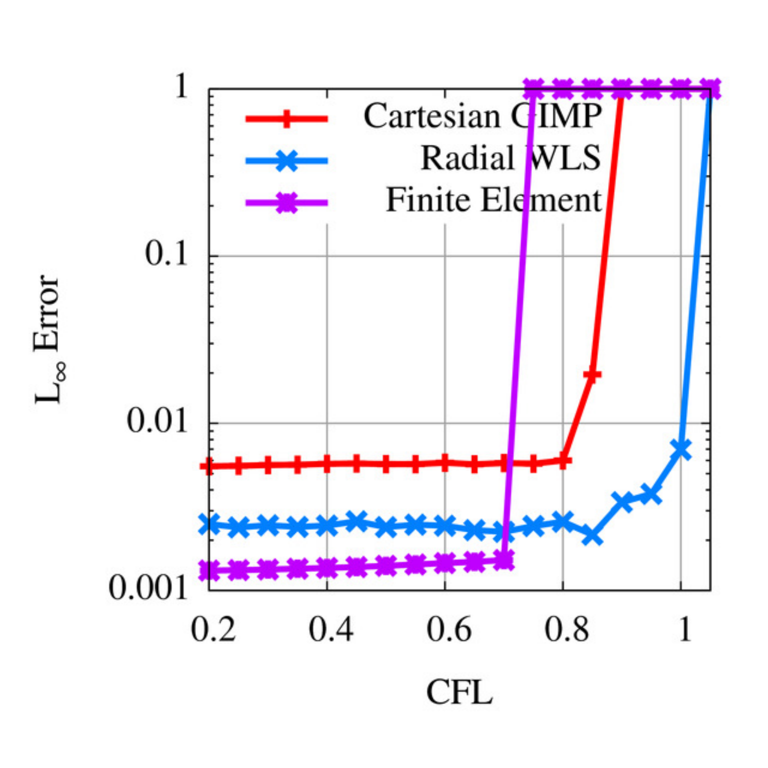
$$f_i^{int} = -\sum_e \sigma(F_e)\nabla\phi_e(x_i)$$

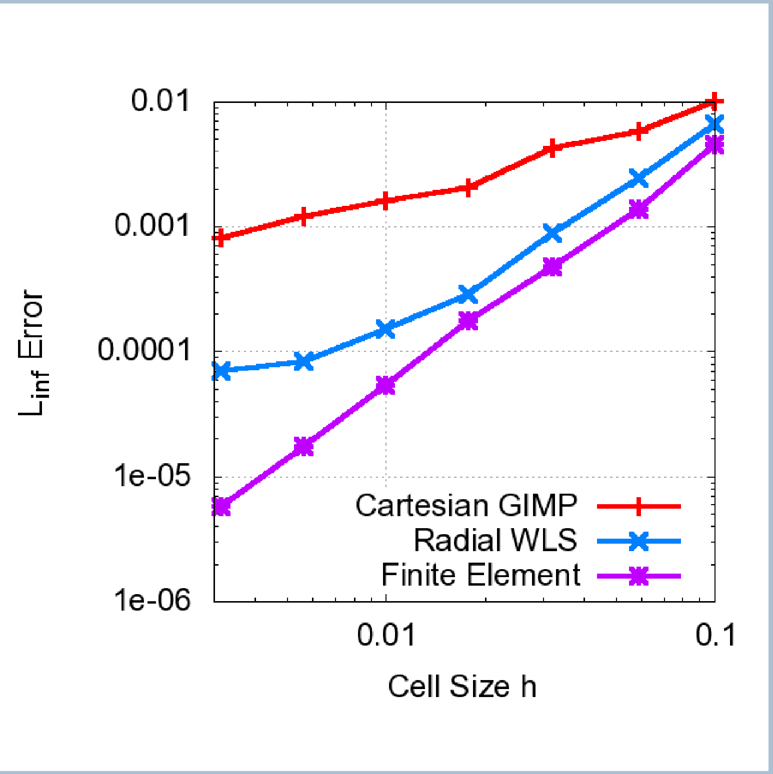
Time update:

$$\mathbf{v}_i^{n+1/2} = \mathbf{v}_i^{n-1/2} + \mathbf{a}_i^n \Delta t$$

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \mathbf{v}_i^{n+1/2} \Delta t$$

Ring





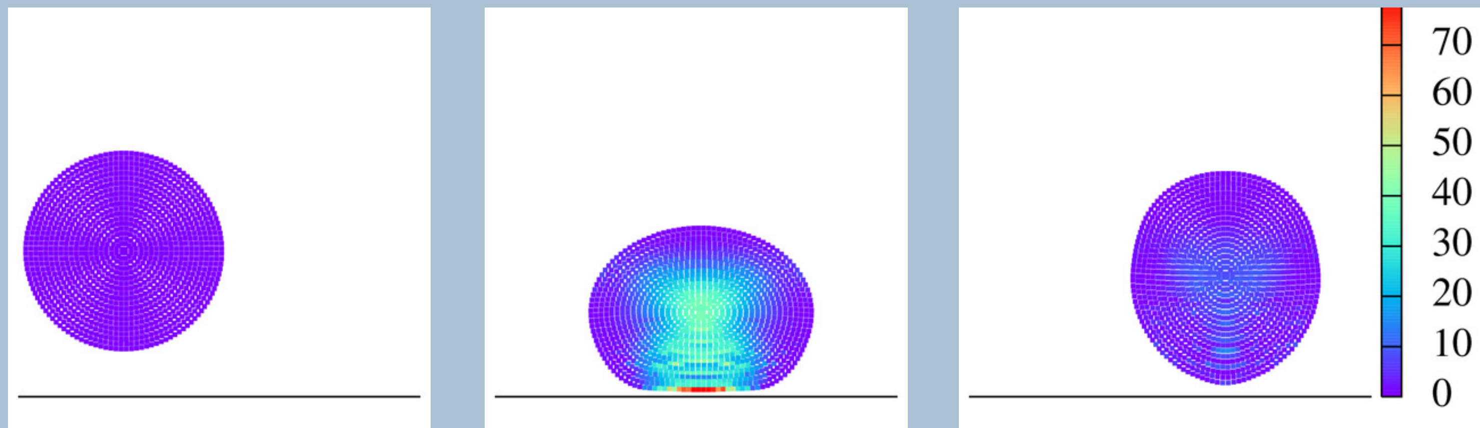
WLS and FEM temporal behavior like GIMP.

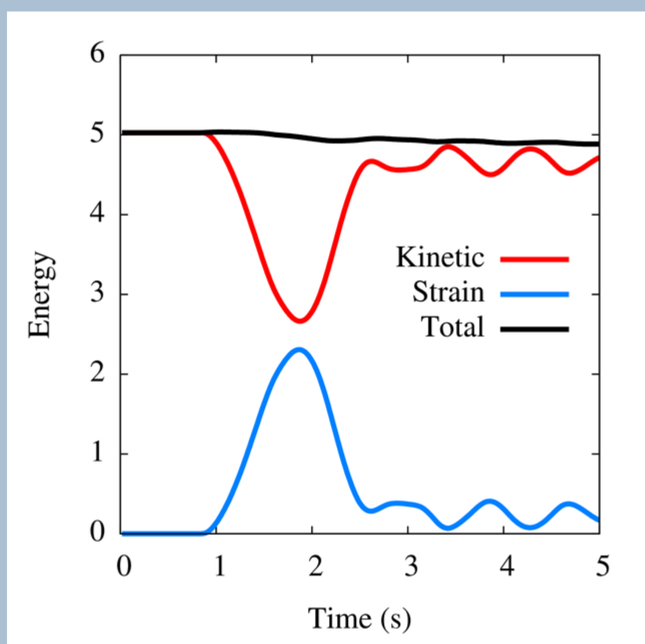
FEM converges "forever".

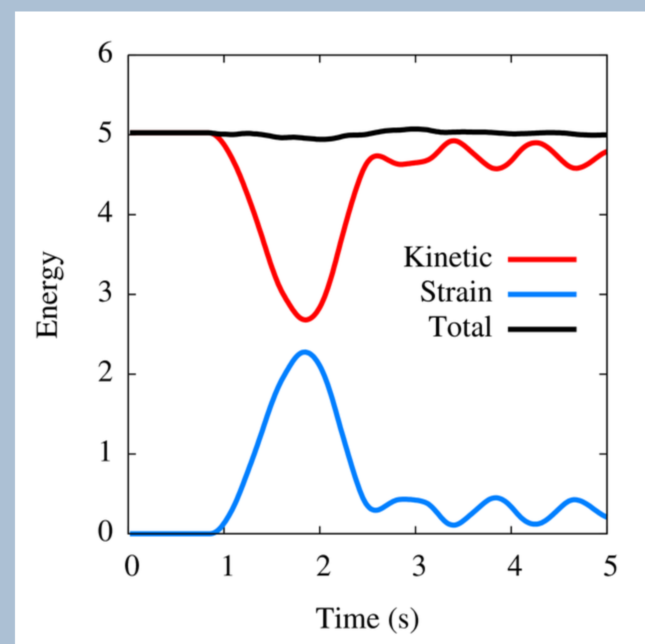
WLS begins 2nd order, but weakens for finer meshes

GIMP is 1st order

Single-ball impact

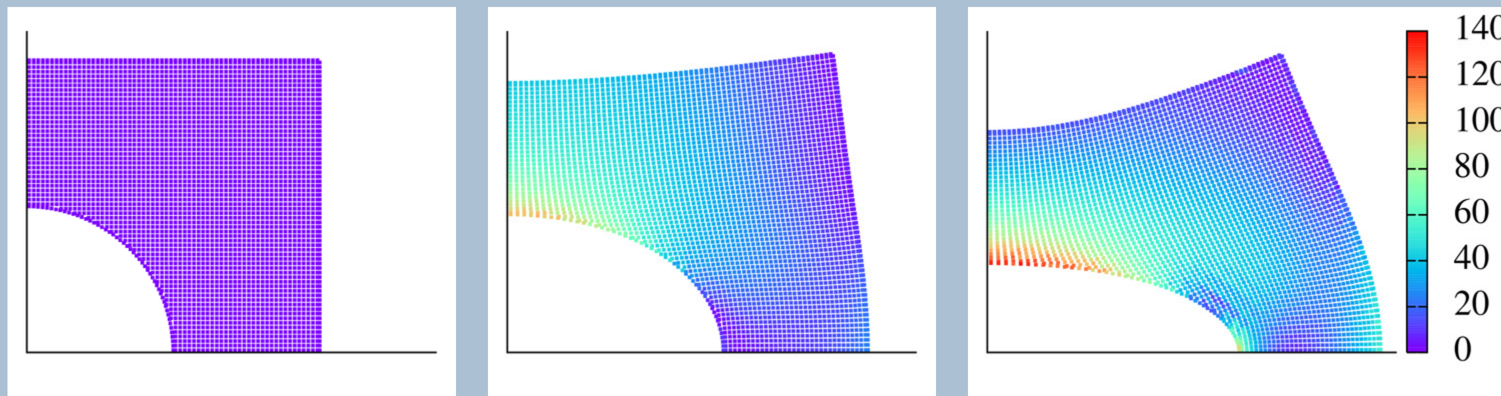






Energy conservation is similar to GIMP

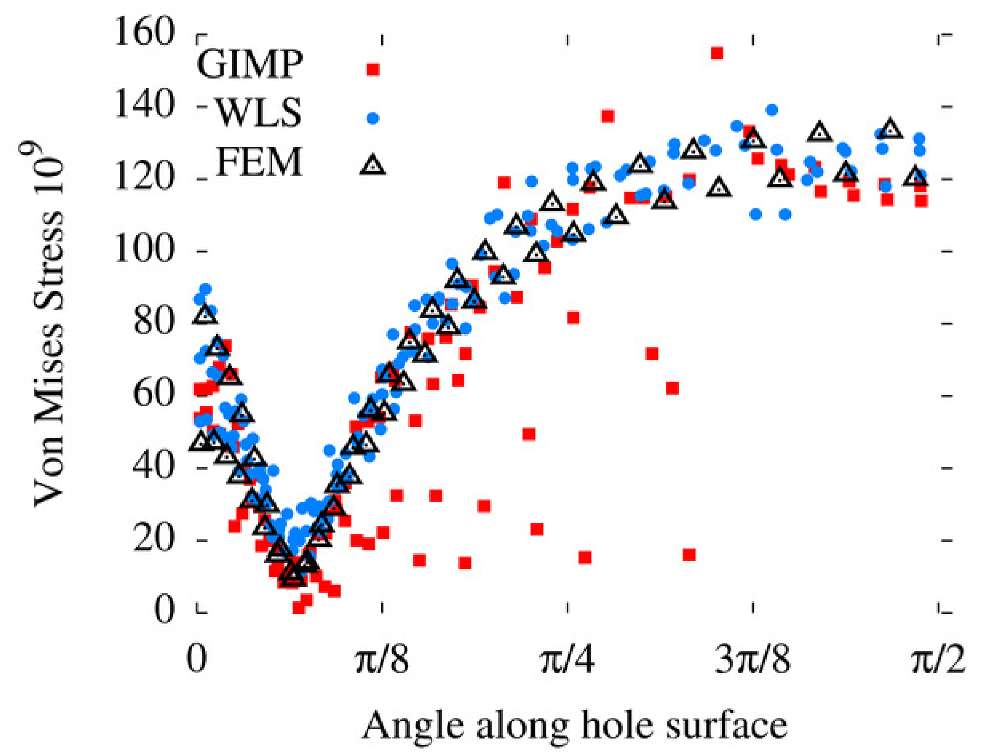
Dynamic hole-in-plate



FEM represents envelope of acceptable accuracy for given cell size.

WLS stays close to FEM accuracy.

GIMP shows large scatter, regardless of mesh size.



Transitioning WLS to GIMP

- WLS interior particles are initialized exactly like GIMP, and contain same information
- WLS bad for rough and porous surfaces like foam, and bad for new surfaces from material failure
- All information is present to transition from WLS to GIMP in mid-problem
- Example: model an over-pressurized tank with WLS to find location and value of highest pressure, then transition to GIMP and continue modeling tank rupture

Conclusions

• Weighted Least Squares is demonstrated to improve accuracy while staying within PIC framework.

• Implicit surface and marching squares can be used to form regions of integration, but implementation is complicated.

• Ill-conditioned nodes are significant limitation of PIC.

• GIMP is somewhat faster than WLS, slower than FEM, first order demonstrated accuracy, but formal order is unclear.